EULER-MARUYAMA'S METHOD FOR NUMERICAL SOLUTION OF ORNSTEIN-UHLENBECK'S EQUATION

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As a partial fulfillment for bachelor degree in science

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Dengan ini menyatakan bahwa skripsi saya dengan judul **"Metode Euler-Maruyama Untuk Solusi Numerik Persamaan *Ornstein-Uhlenbeck*"** adalah benar merupakan hasil karya saya dan bukan merupakan plagiat dari karya orang lain atau pengutipan dengan cara-cara yang tidak sesuai dengan etika yang berlaku dalam tradisi keilmuan. Apabila suatu saat terbukti saya melakukan plagiat maka saya bersedia diproses dan menerima sanksi akademis maupun hukum dan ketentuan yang berlaku, baik di institusi UNP maupun di masyarakat dan Negara.

Demikianlah pertanyaan ini saya buat dengan penuh kesadaran dan rasa tanggung jawab sebagai anggota masyarakat ilmiah.

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Euler-Maruyama's Method for Numerical Solution of Ornstein-Uhlenbeck's Equation

Elga Sania Rahman

ABSTRACT

The Ornstein-Uhlenbeck's equation is a stochastic differential equation which is frequently used in financial mathematical models. However, it is almost impossible to find the solution the Ornstein-Uhlenbeck's equation analytically and hence a numerical solution becomes the best alternative. A good numerical method is considered by examining its convergence speed. The purposes of the study are to examine if the Euler-Maruyama's method can provide a numerical solution for the Ornstein-Uhlenbeck's equation with a strong convergence to the exact solutions and to build an algorithm that represent the Euler-Maruyama's method in solving the Ornstein-Uhlenbeck equation.

The research is classified as a basic (theoretical) research since it discusses the theories on Ornstein-Uhlenbeck's equation. The method used in this study was the Euler-Maruyama method. This Euler-Maruyama's method is derived from a stochastic differential equation generalized from the Euler's method for ordinary differential equations for stochastic differential equations.

In this study, a formula of Euler-Maruyama namely $X(t_{n+1}) = X(t_n) + \lambda\big(\mu - X(t_n)\big)h + \sigma(W(t_{n+1}) - W(t_n))$, with $n = 0, \dots, N$ is used to yield $E[|e_N|^2] \leq Ch$. Using the definition of convergence, the solution by Euler-Maruyama method for Ornstein-Uhlenbeck's equation has a strong convergence. The Euler-Maruyama method algorithm starts by entering the function $f(X(t)) = \lambda\big(\mu - X(t)\big)$ and $g\big(X(t)\big) = \sigma$, by discretizing the interval $[0, T]$ into N intervals of width $h = \frac{T}{N}$, dan X_0 (initial value). The algorithm is implemented using loop calculation to obtain approximate solution for the Ornstein-Uhlenbeck's equation.

Keywords: Stochastic Differential Equations, Ornstein-Uhlenbeck Equation, Euler-Maruyama's Method.

FOREWORD

بسم الله الرحمن الرحيم

Alhamdulillahi rabbil 'alamin all praise to be with Allah SWT for all His mercy and blessing so I can complete the writing of the thesis entitled **"Euler-Maruyama's Method for Numerical Solution of Ornstein-Uhlenbeck's Equation".** Salawat and salam to be with the Prophet Muhammad S.A.W the prominent role model for all mankind.

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Padang, 12 February 2020

Author

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CHAPTER I

INTRODUCTION

A. Research background

The Ornstein-Uhlenbeck's equation represents a stochastic process given by the stochastic differential equation, generally implemented in the problems of financial mathematics and physics. The stochastic differential equation is a model for answering problems whose distribution changes by time, thus classified as a stochastic process. A stochastic differential equation can be obtained by adding a random disturbance term to the deterministic differential equation. A random disturbance is called Brownian motion or Wiener process.

In history, the oldest examples of these equations have been used to describe the motion of particles affected by friction. This equation is named after Leonard Ornstein and George Eugene Uhlenbeck (Mao, 2010). Examples of models using Ornstein-Uhlenbeck's equation include stock price movements and wind movements. Bank interest rates and the level of progress of a company's business fluctuate erratically causing stock prices follow a stochastic process since their values vary in unexpected patterns. The behavior of stock price movements can be estimated using a model of stock price movements associated with a stochastic differential equation (SDE). Another example of this stochastic differential equation is a model of wind movement as it is affected by air pressure and wind speed.

In general, an Ornstein-Uhlenbeck's equation is stated as:

$$dX(t) = \lambda\big(\mu - X(t)\big)dt + \sigma dW(t)$$

In this model, $\lambda\big(\mu - X(t)\big)dt$ is the deterministic term and $\sigma dW(t)$ is the diffusion term. The diffusion term of this Ornstein-Uhlenbeck's equation is independent of $X(t)$, therefore $X(t)$ can be either positive or negative. The equation above has an explicit solution, but it will be difficult to find a solution because this explicit solution still contains stochastic elements, hence stochastic calculus is required to solve such equation.

The solution of Ornstein-Uhlenbeck's equation is difficult to find analytically but it is possible to find the solution with numerical methods. The numerical method is an alternative to the analytic method when it is impossible to solve a mathematical problem using analytical method. However, with the numerical method the solution is not exactly accurate since the method only gives an approach to the exact solution, hence the solution is also called an approximate solution. The difference between the approximate solution and the exact solution is called the error.

A number of numerical methods that can be used to approximate the solution of a stochastic differential equation have been developed along with typical convergence properties, including: the Euler-Maruyama method, the Euler-Milstein method, the implicit method and the explicit method. Most of the numerical methods of stochastic differential equations are derived from the Itô Taylor expansion.

The Euler-Maruyama method is an extension of the Euler method for deterministic differential equations and stochastic differential equations, named after

Leonhard Euler and Gisiro Maruyama. This method is derived from Itô Taylor's expansion by taking the first three terms of Taylor's series. Meanwhile the Euler-Milstein Method is a continuation of the Euler-Maruyama method by involving other terms with higher orders. In Ornstein-Uhlenbeck equation, the function in the deterministic term is a constant which makes its derivative is zero. Therefore, the Euler-Maruyama method is considered to be more appropriate for solving Ornstein-Uhlenbeck's equation numerically.

To find out if the numerical method gives the desired results, it is necessary to perform a numerical analysis to test the accuracy of the numerical method used. The main aspect to measure is the convergence speed, i.e how fast the numerical calculation approaches the exact solution given a tolerated error value. To prove if the Ornstein-Uhlenbeck numerical solution is close to the exact solution with an acceptable error percentage (strong convergence), the deterministic term and the diffusion term must satisfy the local conditions of Lipschitz's.

Based on the description above, it is necessary to present the Euler-Maruyama method to solve the Ornstein-Uhlenbeck's equation in a thesis entitled as "**Euler-Maruyama Method for Numerical Solution of the Ornstein-Uhlenbeck Equation**".

B. Problem formulation

Based on the research background presented above, the problem of the research can be formulated within a question: *What is the Euler-Maruyama method for the numerical solution of the Ornstein-Uhlenbeck equation*?

C. Research questions

To represent the problem discussed in this thesis, the following questions are needed to be answered:

1. How is the formula of Euler-Maruyama method in finding the numerical solution of *Ornstein-Uhlenbeck* equation?
2. How is the convergence of Euler-Maruyama method in finding the numerical solution of *Ornstein-Uhlenbeck* equation?
3. How is the algorithm representing the Euler-Maruyama method in finding the numerical solution of *Ornstein-Uhlenbeck* equation?

D. Research objectives

Research objectives are to answer the research questions that have been presented above:

1. Analyzing the formula of Euler-Maruyama method in finding the numerical solution of *Ornstein-Uhlenbeck* equation.
2. Analyzing the convergence of Euler-Maruyama method in finding the numerical solution of *Ornstein-Uhlenbeck* equation.
3. Building the formula of Euler-Maruyama method in finding the numerical solution of *Ornstein-Uhlenbeck* equation.

E. Research benefits

This research is expected to give the following benefits:

1. To provide additional insight and knowledge for researchers and readers about convergence analysis, especially in numerical methods for solving stochastic differential equations.
2. As an input for further research in developing and expanding the scope of research.
3. As a supportive learning material for students in the field of numerical analysis, especially numerical methods for the Ornstein-Uhlenbeck equation.

F. Research methods

This research is a basic (theoretical) research that analyzes relevant theories concerning the problems discussed based on literature review. To solve the research problem, the following steps are taken:

1. Studying the literatures on stochastic differential equations and numerical methods.
2. Discussing the principles of the usage of Euler-Maruyama method to solve stochastic differential equations.
3. Proving the convergence of the Euler-Maruyama method in finding the solution of the Ornstein-Uhlenbeck equation.
4. Developing algorithms to represent the Euler-Maruyama method in a computer program to solve the Ornstein-Uhlenbeck equation.
5. Drawing the research summaries.

CHAPTER II

THEORETICAL REVIEW

A. Lipschitz's Function

Definition:

Let f is defined on D, where D is a closed region in xy. The function $f$is said to satisfy Lipschitz's constant on D if exists a constant $k > 0$ such that:

$$|f(x) - f(y)| \leq K|x - y|$$

for all x and y in D. The constant K is called Lipschitz's constant.

(Shepley,1989)

The Lipschitz condition above is a condition that must be satisfied by a differential equation in order to guarantee there is a single solution for the equation.

B. Brownian motion/ Wiener's process

Brownian motion (Brownian motion) is a term for irregular motion of pollen suspended in water observed by the Scottish botanist Robert Brown in 1828. This motion is later explained by random collisions with water molecules. To describe this movement mathematically, a concept of stochastic process $W_t(\omega)$ is used and interpreted as the position of pollen ω at time t (Mao, 2007).

The definition of Brownian motion according to Mao:

Let $(\Omega, \mathcal{F}, P)$ be the probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. One dimensional Brownian motion is a continuous real value motion $\{\mathcal{F}_t\}$. The application process $\{B_t\}_{t\geq 0}$ has the following properties:

(i) $B_0 = 0$;

(ii) For $0 \leq s < t < \infty$, the increment $B_t - B_s$ is normally distributed with expectation 0 and variance $t - s$

(iii) For $0 \leq s < t < \infty$, the increment $B_t - B_s$ and $\mathcal{F}_s$ are independent

According to Mao (2010) a Brownian motion has some important properties summarized below:

a. $\{-W_t\}$ is a Brownian motion with the same filtration $\{\mathcal{F}_t\}$

b. Let $c > 0$. Define:

$$X_t = \frac{B_{ct}}{\sqrt{c}} \text{ for } t \geq 0$$

then X_t is a Brownian motion with respect to the filtration $\{\mathcal{F}_{ct}\}$

c. $\{W_t\}$ is a continuous square-integrable martingale and its quadratic variation $(W, W_t) = t$ for all $t \geq 0$.

d. $\lim\limits_{t \to \infty} \frac{W_t}{t} = 0$.

C. Stochastic Process

The stochastic process is a series of random variables X_t where $t \in T$ is an index of time. The result space for X_t can be either discrete or continuous (Gallagher, 2013).

The stochastic process is a process of a series of random events defined as the set of random variables X_t where t represents time (Knill, 2009). Random events can be interpreted as states collected in a state space. The state space is the set of possible values for the random variable X_t (Taylor, 1998)

According to Gallagher (2013), a stochastic process $X = \{X_t, t \in T\}$ is defined as a series of random variables where for all $t \in T$ there exists random variable X_t where t represents time. The value of random variable X_t is called the state at time t. The set T is called parameter space or index space of a stochastic process X and each realization of X is called sample path of X.

Platen (2007) defines stochastic process as:

Definition 1: the set $X = \{X_t, t \in T\}$ of random variable $X_t \in \mathbb{R}^d$ is a d dimensional stochastic process, where the total of the dimensional distribution function is finite.

$$F_{X_{t_{i_1}}, \cdots, X_{t_{i_j}}}(x_{i_1}, \cdots,)x_{i_j} = P\ (X_{t_{i_1}} \leq x_{i_1}, \cdots, X_{t_{i_1}} \leq x_{i_j})$$

for $i_j \epsilon \mathcal{N}, x_{i_j} \epsilon \mathbb{R}^d$ and $t_{i_j} \in T$ determines the probability.

where T is a set of time, defined on an interval of $T = [0, \infty)$. In some occasion, the interval is limited to $[0, T]$ for $T \in (0, \infty)$ or a discrete time set $\{t_0, t_1, t_2 \ldots\}$, where $t_0 < t_1 < t_2 < \ldots$

A stochastic process has an expectation of:

$$\mu(t) = E(X_t)$$

and variance

$$v(t) = var\ (X_t) = E((X_t - \mu(t)\ )^2)$$

for $t \geq 0$, with a covariance

$$C(s, t) = cov(X_s, X_t) = E\big((X_s - \mu(s)\big)\big(X_t - \mu(t)\big))$$

for $s, t \in T$.

1. Stochastic integral

Stochastic integral is an integral where the sum is more than an integration and multiplied by the increase in time on the Wiener process trajectory (Dumas and Luciano, 2017).

Mao (2010) defines a stochastic integral as follows:

$$\int_{0}^{t} f(s) dW_s$$

With respect to a Brownian motion or Wiener process $\{W_t\}$ for a stochastic process $f(t)$.

Definition 1. Let $0 \leq a < b < \infty$. $\mathcal{M}^2([\mathrm{a,b}]; \mathrm{R})$ is the space of all real-valued measurable $\{\mathcal{F}_t\}$-adapted process $f = \{f(t)\}_{a \leq t \leq b}$ such that

$$\|f\|_{a,b}^2 - E \int_a^b |f(t)|^2 dt < \infty$$

Definition 2. A real valued stochastic process $g = \{g(t)\}_{a \leq t \leq b}$ is called a simple process if there exists a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$, and bounded random variable $\xi_i, \leq i \leq k - 1$ such that ξ_i is $\mathcal{F}_{t_i}$- measurable and

$$g(t) = \xi_0 I_{[t_0, t_i]}(t) + \sum_{i=1}^{k1} \xi_i I_{[t_i, t_{i+1}]}(t). \quad \text{... (1)}$$

$\mathcal{M}_0([a, b]; \mathcal{R})$ denotes the family of all such processes.

Definition 3 (Itô's integral)

For a simple process g with the form (1), define

$$\int_a^b g(t)dW(t) = \sum_{i=0}^{k-1} \xi_i (W_{t_{i+1}} - W_{t_i})$$

and called as the stochastic integral of g or the Itô's integral.

Lemma 1

If g is a stochastic process, then

$$E\int_a^b g(t)dW(t) = 0$$

$$E\left|\int_a^b g(t)dW(t)\right|^2 = E\int_a^b |g(t)|^2\,dt$$

Proof:

Since ξ_i is $\mathcal{F}_{t_i}$-measurable whereas $W_{t_{i+1}} - W_{t_i}$ is independent of $\mathcal{F}_{t_i}$,

$$E\int_a^b g(t)dW(t) = \sum_{i=0}^{k-1} E[\xi_i(W_{t_{i+1}} - W_{t_i})] = \sum_{i=0}^{k-1} E\xi_i\, E[(W_{t_{i+1}} - W_{t_i})] = 0$$

Note that $W_{t_{j+1}} - W_{t_j}$ is independent of $\xi_i\xi_j(W_{t_{i+1}} - W_{t_i})$ if $i < j$, thus:

$$E\left|\int_a^b g(t)dW(t)\right|^2 = \sum_{0\le i,j\le k-1} E\left[\xi_i\xi_j\left(W_{t_{i+1}} - W_{t_i}\right) - \left(W_{t_{j+1}} - W_{t_j}\right)\right]$$

$$= \zeta\sum_{i=0}^{k-1} E[{\xi_i}^2(W_{t_{i+1}} - W_{t_i})^2] = \sum_{i=0}^{k-1} E{\xi_i}^2(W_{t_{i+1}} - W_{t_i})^2$$

$$= \sum_{i=0}^{k-1} E{\xi_i}^2(t_{i+1} - t_i)^2] = E\int_a^b |g(t)|^2\,dt$$

Lemma 2

Let g_1, g_2 are stochastic processes and let c_1, c_2 be two real numbers, then $c_1g_1 + c_2g_2$ is a stochastic process and

$$\int_a^b [c_1g_1(t) + c_2g_2(t)]\,dW(t) = c_1\int_a^b g_1(t)dW(t) + c_2\int_a^b g_2(t)dW(t)$$

Proof:

$$\int_a^b [c_1g_1(t) + c_2g_2(t)]\,dW(t) = \int_a^b c_1g_1(t)dW(t) + \int_a^b c_2g_2(t)dW(t)$$

$$= c_1\int_a^b g_1(t)dW(t) + c_2\int_a^b g_2(t)dW(t)$$

2. Formula Itô

Definition 1.

A one-dimensional Itô's process is a continuous adapted process $x(t)$ on $t \geq 0$ of the form

$$x(t) = x(0) + \int_0^t f(s)ds + \int_0^t g(s)dB_s$$

where $f \in \mathcal{L}^1(\mathcal{R}_+;\mathcal{R})$ and $g \in \mathcal{L}^2(\mathcal{R}_+;\mathcal{R})$. It is said that the process $x(t)$ to have stochastic differential $dx(t)$ on $t \geq 0$ given by

$$dx(t) = f(t)dt + g(t)dB_t$$

Theorem 1 (one-dimensional Itô's formula)

Let $x(t)$ be an Itô's process on $t \geq 0$ with the stochastic differential

$$dx(t) = f(t)dt + g(t)dB_t$$

where $f \in \mathcal{L}^1(\mathcal{R}_+; \mathcal{R})$ and $g \in \mathcal{L}^2(\mathcal{R}_+; \mathcal{R})$. Let $V \in C^{2.1}(\mathcal{R} \times \mathcal{R}_+; \mathcal{R})$, then $V(x(t), t)$ is also an Itô's process with the stochastic differential given by

$$dV(x(t), t) = \left[V_t(x(t), t) + V_x(x(t), t)f(t) + \frac{1}{2}V_{xx}(x(t), t)g^2(t)\right]dt$$
$$+ V_x(x(t), t)\ \ g(t)dB_t$$

(Mao, 2010)

3. Gronwall's inequality

The integral inequalities of Gronwall type have been widely applied in the theory of ordinary differential equations and stochastic differential equations to prove the results on existence, uniqueness, boundedness, comparison, continuous dependence, perturbation and stability etc

Theorem (Gronwall's Inequality):

Let > 0 and $c \geq 0$. Let u be a nonnegative function on $[0, T]$,and let v be a nonnegative integrable function on $[0, T]$. If

$$u(t) \leq c + \int_0^t v(s)u(s)ds \qquad \forall 0 \leq t \leq T,$$

then

$$u(t) \leq c \exp\left(\int_0^t v(s)ds\right) \quad \forall 0 \leq t \leq T.$$

Proof:

Without loss of generality, assume that $c > 0$. Set

$$z(t) = c + \int_0^t v(s)u(s)ds \quad \text{untuk } 0 \leq t \leq T.$$

then $u(t) \leq z(t)$. By using chain rule of calculus, obtained

$$\log\big(z(t)\big) = \log(c) + \int_0^t \frac{v(s)u(s)}{z(s)} \leq \log(c) + \int_0^t v(s)ds$$

This implies

$$z(t) = c\exp\left(\int_0^t v(s)ds\right) \quad \text{untuk } 0 \leq t \leq T.$$

Theorem (Gronwall's discrete inequality):

Let M be a positifve integer and let u_k and v_k be a non negative series with $k = 0,1,\cdots M$. If:

$$u_k = u_0 + \textstyle\sum_{j=0}^{k-1} v_j u_j \ , \forall k = 1,2,\cdots M, \qquad \text{... (2)}$$

then,

$$u_k \leq u_0 \exp\left(\sum_{j=0}^{k-1} v_j\right), \quad \forall k = 1,2,\cdots M$$

Proof:

Define $t \in [0, M+1)$,

$$u(t) = \sum_{j=0}^{M} u_j I_{[j,j+1)}(t), \qquad v(t) = \sum_{j=0}^{M} v_j I_{[j,j+1)}(t),$$

and

$$A(t) = \sum_{j=0}^{M} I_{[j,j+1)}(t)$$

According to (2) it implies that

$$u(t) \leq u(0) + \int_0^t v(s)u(s-)dA(s), \quad \forall t \in [0,T]$$

where the integral is a type of Lebesgue-Stieltjes and $u(s-) = \lim_{t \uparrow s} u(t)$(left limit). Assume that $u(0) = u_0 > 0$, if not subtitute u_0 by $\varepsilon > 0$ and let $\varepsilon \to 0$. Then

$$w(t) = u(0) + \int_0^t v(s)u(s-)dA(s), \quad \forall t \in [0, M+1)$$

where $w(t)$ is positive and non-decreasing, and $w(t) \geq u(t)$ where $w(t-) \geq u(t-)$. Using basic differential formula, obtained

$$\log(w(t)) = \log(u(0)) + \int_0^t \frac{v(s)u(s-)}{w(s-)} dA(s)$$

$$+ \sum_{0<s\leq t} \left[\log(w(s)) - \log(w(s-)) - \frac{w(s)w(s-)}{w(s-)}\right] \qquad \text{... (3)}$$

However, the following holds

$$\log(w(s)) - \log(w(s-)) = \int_{w(s)}^{w(s-)} \frac{dz}{z} \leq \int_{w(s)}^{w(s-)} \frac{dz}{w(s-)} \leq \frac{w(s)w(s-)}{w(s-)}$$

Subtitute the above equation into Eq. (3) and by using $w(s-) \geq u(s-)$, obtained:

$$\log(w(t)) = \log(u(0)) + \int_0^t v(s)dA(s)$$

for all $k = 1,2, \cdots M$, Now we have

$$\log\big(w(k)\big) \leq \log(u_0) + \int_0^k v(s)dA(s) = log(u_0) + \sum_{j=0}^{k-1} v_j$$

Which equals to

$$u_k \leq \mathrm{w}(k) \leq u_0 \exp\left(\sum_{j=0}^{k-1} v_j\right)$$

(Mao dan Yuan, 2006)

D. Taylor's series

Taylor series is a series of polynomials which is often used to solve differential equations. Some theorems are presented below regarding the Taylor series:

Theorem:

If f and its derivatives, $f', f'', f''', \ldots$ are continuous on $[a, b]$, then for the values of x close to x_0 and $x \in [a, b]$, $f(x)$ is expandable into a Taylor's series:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \ldots$$

$$+ \qquad \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Let $x_0 = a$ then the above equation can be written as follows:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots$$

$$+ \qquad \frac{f^{(n)}(a)}{n!}(x - a)^n$$

(Munir, 2008: 361)

E. Stochastic differential equation

A process is said to be a diffusion process if it is a Markov process in which changes in an event occur over time continuously (Cox dan Miller, 1980).

According to Sobczyk (1990), a Markov's process $X(t), t \subseteq T$ whose value is in R_1 is called a diffusion process if the changes in the probability of distribution function on a very little time interval $(\triangle t)$ of an event of process carries on some small changes of state.

The diffusion process in general is expressed as a stochastic differential equation below:

$$dX(t) = \mu \, dt + \sigma \, dW(t)$$

where;

$\mu \, dt$: deterministic component

$\sigma \, dW(t)$: stochastic component (noice), $W(t)$ is the Wiener's process, $dW(t)$ has the $dX(t) = 0$ and variance of 1.

For a small interval of time t, denoted by $\triangle t$ then $\triangle X(t) = \mu \triangle t + \sigma\varepsilon\sqrt{\triangle t} \sim N(\mu \triangle t, \sigma^2 \triangle t)$ such that $(T) - X_0 \sim N(\mu \triangle t, \sigma^2 T)$ $dX(t)$, has a drift rate of μ and rate of σ^2. If μ and σ are not constants but functions then it forms Ito's stochastic differential equations that can be written as:

$$dX(t) = \mu \left(t, X(t)\right) dt + \sigma\left(t, X(t)\right) dW(t)$$

One example of a stochastic differential equation is:

Ornstein-Uhlenbeck's Stochastic Differential Equation

In mathematics, the Ornstein-Uhlenbeck process is a stochastic process commonly applied in financial mathematics and physical science. It was originated in physics presented as a model for the velocity of large Brown particles affected by friction. This model is named after Leonard Ornstein and George Eugene Uhlenbeck. According to (Thierfelder, 2015) the Ornstein-Uhlenbeck differential equation is obtained from the stochastic process with the equation:

$$dX(t) = \lambda\big(\mu - X(t)\big)dt + \sigma dW(t)$$

This model depends on λ which indicates the stability of the model. The model is considered to be stable if $\lambda > 0$. The solution for this equation can be found by proofing Itô's lemma using $e^{\lambda t}X(t)$ as follows:

Solution:

$$dX(t) = \lambda\big(\mu - X(t)\big)dt + \sigma dW(t)$$

$$dX(t) = \lambda\mu dt - \lambda X(t)dt + \sigma dW(t)$$

$$dX(t) + \lambda X(t)dt = \lambda\mu dt + \sigma dW(t)$$

$$e^{\lambda t}dX(t) + e^{\lambda t}X(t)dt = \lambda\mu e^{\lambda t}dt + \sigma e^{\lambda t}dW(t)$$

$$d\left(e^{\lambda t}X(t)\right) = \lambda\mu e^{\lambda t}dt + \sigma e^{\lambda t}dW(t)$$

with integral with respect to time

$$\int_0^t d\left(e^{\lambda t}X(t)\right) = \int_0^t \lambda\mu e^{\lambda t}dt + \int_0^t \sigma e^{\lambda t}dW(t)$$

$$e^{\lambda t}X(t) - e^{0}X(0) = \lambda\mu\frac{e^{\lambda t} - e^{0}}{\lambda} + \sigma e^{-\lambda t}\int_{0}^{t} e^{\lambda t}\, dW(t)$$

$$X(t) - X(0)e^{-\lambda t} = \mu\left(1 - e^{-\lambda t}\right) + \sigma e^{-\lambda t}\int_{0}^{t} e^{\lambda s}\, dW(s)$$

$$X(t) = \ X(0)e^{-\lambda t} + \ \mu\left(1 - e^{-\lambda t}\right) + \sigma e^{-\lambda t}\int_{0}^{t} e^{\lambda s}\, dW(s)$$

where

$$X(t) = \mu + e^{-\lambda t}(\ X(0) - \mu) + \ \sigma e^{-\lambda t}\int_{0}^{t} e^{\lambda s}\, dW(s)$$

For initial condition $X(0) = X(0)$

$$E\big(X(t)\big) = E\left[\mu + e^{-\lambda t}(\ X(0) - \mu) + \ \sigma e^{-\lambda t}\int_{0}^{t} e^{\lambda s}\, dW(s)\right]$$

$$= E\big[\mu + e^{-\lambda t}(\ X(0) - \mu)\big] + E\left[\sigma e^{-\lambda t}\int_{0}^{t} e^{\lambda s}\, dW(s)\right]$$

$$= \mu + e^{-\lambda t}(\ X(0) - \mu)$$

$var\big(X(t)\big) = E[X(t) - E[X(t)]^2]$

$$= E\left[\left(\mu + e^{-\lambda t}(\ X(0) - \mu) + \sigma e^{-\lambda t}\int_{0}^{t} e^{\lambda t}\, dW(s) - \mu + e^{-\lambda t}(\ X(0) - \mu)\right)^2\right]$$

$$= E\left[\left(\sigma e^{-\lambda t}\int_{0}^{t} e^{\lambda s}\, dW(s)\right)^2\right]$$

$$= \sigma^2\int_{0}^{t} e^{-2\lambda(t-s)} ds$$

$$= \sigma^2\,\frac{e^{0} - e^{-2\lambda t}}{2\lambda}$$

F. Euler-Maruyama's Method

The numerical method is a technique used to formulate mathematical problems in order to be able to solve using ordinary operations. In this study the numerical method used for solving stochastic differential equations is the Euler-Maruyama method which simply generalized from Euler's stochastic differential equations.

According to (Fadugba,2013) the Euler-Maruyama's method is an analog of Euler's method to solve an ordinary differential equation represented by Taylor's series thus the analysis is simplified. Let $X(t)$ be an Ito's process at $t \in [t(0), T]$ and a stochastic differential equation:

$$dX(t) = f\big(t, X(t)\big)dt + g\big(t, X(t)\big)dW(t) \qquad \ldots (1)$$

$$X\big(t(0)\big) = X(0)$$

is given in its discrete form $t_0 < t_1 < t_2 < \cdots < t_N = T$, with respect to continuous time such that it can be written as follows:

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} f(t, x)dt + \int_{t_n}^{t_{n+1}} g(t, x)dW(t) \qquad \ldots (2)$$

The first step taken to solve the equation above is substituting Eq. (1) to the integral equation (2) by expanding Taylor's series with the points(t_n, x_n) and $X(n) = X(t)$. The term $(x - x(n))$ in the expansion performed above can be rewritten as:

$$X(t) = X(t_n) + \int_{t_n}^{t_1} f(t', x)dt' + \int_{t_n}^{t} g(t', x)dW(t') \qquad \ldots (3)$$

Express in a Taylor's series given the functions $f(t, x)$ and $g(t, x)$ above,

$$f(t,x) = f(t_n + \delta t, x(n) + \delta x) = f\big(t_n, x(n)\big) + \tfrac{\partial f}{\partial t}\delta t + \tfrac{\partial f}{\partial x}\partial x + \cdots$$

$$= f\big(t_n, x(n)\big)\tfrac{\partial f}{\partial t}\bigg|_{(t_n - x_n)}^{t - t_n} + \tfrac{\partial f}{\partial x}\bigg|_{(t_n - x_n)}^{x - x(n)} + \cdots \qquad \ldots (4)$$

$$g(t,x) = (t_n + \delta t, x(n) + \delta x) = f\big(t_n, x(n)\big) + \frac{\partial g}{\partial t}\delta t + \frac{\partial g}{\partial x}\partial x + \cdots$$

$$= f\big(t_n, x(n)\big)\tfrac{\partial g}{\partial t}\bigg|_{(t_n - x_n)}^{t - t_n} + \tfrac{\partial g}{\partial x}\bigg|_{(t_n - x_n)}^{x - x(n)} + \cdots \qquad \ldots (5)$$

Substituting Eq. (4) and (5) into Eq. (2):

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}}\left(f(t_n, x_n) + \tfrac{\partial f}{\partial t}\bigg|_{(t_n - x_n)}^{t - t_n} + \tfrac{\partial f}{\partial x}\bigg|_{(t_n - x_n)}^{x - x(n)}\right)dt +$$
$$\int_{t_n}^{t_{n+1}}\left(g(t_n, x_n) + \tfrac{\partial g}{\partial t}\bigg|_{(t_n - x_n)}^{t - t_n} + \tfrac{\partial g}{\partial x}\bigg|_{(t_n - x_n)}^{x - x(n)}\right)dW(t) + \cdots$$

$$\ldots (6)$$

$$X(t_{n+1}) = X(t_n) + f_n(t_{n+1} - t_n) + g_n(W(t_{n+1}) - W(t_n)) \qquad \ldots (7)$$

The equation (7) is called Euler-Maruyama's method.

G. Convergence of the method

In analyzing a numerical method, the analysis is focused on error and the convergence speed of the method.

1. Strong convergence

Definition: A sequence $Y^h = (Y(t))_{t \in \mathcal{I}_h}$ converges p to the solution X of a stochastic differential equation at time T if there exist two constants $C > 0$ and $\delta_0 > 0$ such that for all $h \in [0, \delta_0]$,

$$\left(E(\|X_T - Y^h(T)\|^2)\right)^{1/2} \leq Ch^p$$

2. Weak convergence

Definition: A sequence $Y^h = (Y(t))_{t \in \mathcal{I}_h}$ weakly converges p to the solution X of a stochastic differential equation at time T if for all $f \in C_p^{2(p+1}(R^d, R)$ there exist two C_f and $\delta_0 > 0$ such that for all $h \in [0, \delta_0]$,

$$\left|E\left(f(X_T)\right) - E\left(f\left(Y^h(T)\right)\right)\right| \leq C_f h^p$$

(Rößler,2010)

CHAPTER III

RESULTS AND DISCUSSION

A. Euler-Maruyama's method for Ornstein-Uhlenbeck's equation

Most of numerical methods for solving stochastic differential equation (SDE) is derived from Taylor's series. Below is given an Ornstein-Uhlenbeck's SDE:

$$dX(t) = \lambda\big(\mu - X(t)\big)dt + \sigma dW(t), \quad t\epsilon[0,T] \qquad \text{... (1)}$$

In general, the equation above can be stated in a model form as follows:

$$dX(t) = f\big(X(t)\big) + g\big(X(t)\big)dW(t), \quad t \in [0,T] \qquad \text{... (2)}$$

and stated in an integral form:

$$\int_{t_0}^{t} dX(t) = \int_{t_0}^{t} f\big(X(t)\big)dt + \int_{t_0}^{t} g\big(X(t)\big)dW(t)$$

$$X(t) - X(t_0) = \int_{t_0}^{t} f\big(X(t)\big)dt + \int_{t_0}^{t} g\big(X(t)\big)dW(t)$$

$$X(t) = X(t_0) + \int_{t_0}^{t} f\big(X(t)\big)dt + \int_{t_0}^{t} g\big(X(t)\big)dW(t)$$

Given the discrete form $t_0 < t_1 < t_2 < \cdots < t_N = T$, with respect to continuous time thus can be restated as follows:

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} f(X(t))dt + \int_{t_n}^{t_{n+1}} g\big(X(t)\big)dW(t) \qquad \text{... (3)}$$

For $s \in [t_n, t_{n+1}]$, the Taylor's expansion for f and g around $X(t_n)$ is used to obtain:

$$f\big(X(s)\big) = f\big(X(t_n)\big) + f'(X^*)\big(X(s) - X(t_n)\big) + \cdots \qquad \text{... (4)}$$

$$g\big(X(s)\big) = g\big(X(t_n)\big) + g'(X^{**})\big(X(s) - X(t_n)\big) + \cdots \qquad \text{... (5)}$$

where $X^*, X^{**} \in [X(t_n) - X(t_{n+1})]$

Furthermore, substituting (4) and (5) into Eq. (3), obtained:

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} f(X(t))dt + \int_{t_n}^{t_{n+1}} g\big(X(t)\big)dW(t)$$

$$X(t_{n+1}) = X(t_n)$$

$$+ \int_{t_n}^{t_{n+1}} \Big(f\big(X(t_n)\big) + f'(X^*)\big(X(s) - X(t_n)\big) + \cdots \Big)\ dt$$

$$+ \int_{t_n}^{t_{n+1}} \Big(g\big(X(t_n)\big) + g'(X^{**})\big(X(s) - X(t_n)\big) + \cdots\Big)dW(t)$$

$$X(t_{n+1}) = X(t_n)$$

$$+ \int_{t_n}^{t_{n+1}} f\big(X(t_n)\big)dt + \int_{t_n}^{t_{n+1}} \Big(f'(X^*)\big(X(s) - X(t_n)\big) + \cdots\Big)\, dt$$

$$+ \int_{t_n}^{t_{n+1}} g\big(X(t_n)\big)dW(t)$$

$$+ \int_{t_n}^{t_{n+1}} \Big(g'(X^{**})\big(X(s) - X(t_n)\big) + \cdots\Big)\, dW(t)$$

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} f\big(X(t_n)\big)dt + \int_{t_n}^{t_{n+1}} g\big(X(t_n)\big)dW(t) + \int_{t_n}^{t_{n+1}} \big(f'(X^*)\big(X(s) - X(t_n)\big) + \cdots\big)\, dt + \int_{t_n}^{t_{n+1}} \big(g'(X^{**})\big(X(s) - X(t_n)\big) + \cdots\big)\, dW(t)$$

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} f\big(X(t_n)\big)dt + \int_{t_n}^{t_{n+1}} g\big(X(t_n)\big)dW(t) + R$$

The first three terms of the equation above forms the Euler-Maruyama' s method and R contains the remaining high ordered terms. Thus, the Euler-Maruyama's method for the solution of Ornstein-Uhlenbeck's equation is:

$$X(t_{n+1}) = X(t_n) + f\big(X(t_n)\big)(t_{n+1} - t_n) + g\big(X(t_n)\big)(W(t_{n+1}) - W(t_n)),$$

where $n = 0, \dots, N$, $f\big(X(t_n)\big) = \lambda\big(\mu - X(t_n)\big)$ dan $g\big(X(t_n)\big) = \sigma$

B. Convergence of Euler –Maruyama's (E-M) method for Ornstein-Uhlenbeck's equation

Given the Ornstein-Uhlenbeck's SDE as follows:

$$dX(t) = \lambda\big(\mu - X(t)\big)dt + \sigma dW(t), \quad t\epsilon[0, T] \qquad \dots (1)$$

which can be written in a general model form:

$$dX(t) = f\big(X(t)\big)dt + g\big(X(t)\big)dW(t), \quad t \in [0, T] \qquad \dots (2)$$

$E - M$ method discretizes $[0, T]$ into N intervals, with a width of $h = \frac{T}{N}$

$$0 = t_0, \quad t_1 , \quad t_2 , \dots, \quad t_{N-1}, \quad t_N = T$$

$$t_{n-1} - t_n = h \qquad\qquad n = 0, \dots, N-1$$

Definition 1:

A numerical method for solving a SDE is said to have weak convergence for $\gamma > 0$, if for all p in several classes there exists a constant $C > 0$ such that:

$$\left| E\left[p(X_n) - E\left[p\big(X(\tau)\big) \right] \right] \right| \leq C \Delta t^{\gamma}$$

For all $\tau = n\Delta t \in [0, T]$ and for small Δt.

Definition 2:

A numerical method for solving a SDE is said to have strong convergence on $\mathcal{L}^m$ for $\gamma > 0$, if for $m \geq 1$ there exists a constant $C = C(m) > 0$ such that:

$$E[X_n - X(\tau)^m] \leq C \Delta t^{\gamma m}$$

For all $\tau = n\Delta t \in [0, T]$ and for small Δt.

For $m = 1$ and $m = 2$, there exists $C_1, C_2 > 0$ where:

$E[|X_n - X(\tau)|] \leq C_1 \Delta t^{\gamma}$ and

$E[|X_n - X(\tau)|^2] \leq C_1 \Delta t^{2\gamma}$, $\forall \tau = n\Delta t \in [0, T]$

$E - M$ method has weak convergence with an order of $\gamma = 1$ on the condition that f and g are two global Lipschitz's functions.

Lemma 1:

Let $(X(t))_{t\geq 0}$ is the solution for (2). For each $t, s \in [0, T]$,

$$E[|X(s) - X(t)|^2] \leq 3E\left|\int_t^s f(X(r))dr\right| + 3E\left|\int_t^s g(X(r))dW_r\right|$$

Proof:

$$X(s) - X(t) = \int_t^s f(X(r))dr + \int_t^s g(X(r))dW_r$$

Thus,

$$|X(s) - X(t)| = \left|\int_t^s f(X(r))dr + \int_t^s g(X(r))dW_r\right|^2$$

$$\leq 3\left|\int_t^s f(X(r))dr\right|^2 + 3\left|\int_t^s g(X(r))dW_r\right|^2$$

This uses elementary inequality $(a + b)^2 \leq 3a^2 + 3b^2$ then by taking the expectation from both hands the proof is complete.

Lemma 2:

If f is a global Lipschitz's function then $|f(x)|^2 \leq \widetilde{K}(1 + |x|^2)$ for a $\widetilde{K} > 0, \forall x \in \mathbb{R}$

Proof:

Since f is a global Lipschitz's function, then:

$$|f(x) - f(y)| \leq \widetilde{K}|x - y|$$

$$\Longrightarrow |f(x) - f(0)| \leq \widetilde{K}|x - 0|$$

$$\Longrightarrow |f(x) - f(0)| \leq \widetilde{K}|x| \text{ , } \forall x \in \mathbb{R}$$

By using reverse triangle inequality,

$$|f(x)| - |f(0)| \leq \big||f(x)| - |f(0)|\big|$$

$$\leq |f(x) - f(0)|$$

$$< \widetilde{K}|x|$$

$$\Longrightarrow \ |f(x)| \leq \widetilde{K}|x| + |f(0)|$$

$$\leq K'|x| + K'$$

$$\leq (1 + |x|)$$

where $K' = \max\left(\widetilde{K}, |f(0)|\right)$. Thus,

$$|f(0)|^2 \leq {K'}^2(1 + |x|)^2$$

$$\leq {K'}^2(3(1)^2 + |x|^2)$$

$$\leq 3{K'}^2(1 + |x|^2) \text{ , } \quad \widetilde{K} = 3{K'}^2$$

Theorem 1:

Let $(X(t))_{t\geq 0}$ is the solution (2). E-M method:

$$y_{n+1} = y_n + hf(y_n) + \Delta W_{n+1} g(y_n) \qquad ,n = 0, \dots, N \qquad \dots (3)$$

has strong convergence with $\gamma = \frac{1}{2}$ for $X(t)$ in the case that f and g are two global Lipschitz's functions, there exists a $C > 0$ such that $E[|e_N|^2] \leq Ch$ for h that is small enough.

Proof for Lipschitz's function:

1. $f(x) = \lambda(\mu - x)$

   It will be shown that: $|f(x) - f(y)| \leq K|x - y|$

   Proof: $|f(x) - f(y)| = |\lambda(\mu - x) - \lambda(\mu - y)|$

   $= |-\lambda(x - y)|$

   $= \lambda|(x - y)|$

   $\leq \lambda|(x - y)|$

   Therefore, since $\lambda > 0$ then f satisfies the criteria of a Lipschitz's function.

2. $g(x) = \sigma$

   It will be shown that: $|f(x) - f(y)| \leq K|x - y|$

   Proof: $|f(x) - f(y)| = |\sigma - \sigma|$

   $= 0 < \sigma|x - y|$

   Therefore, by using constant $K = 1$ then g satisfies the criteria of a Lipschitz's function since $\sigma > 0$.

Proof:

For each $n \in \{0, \cdots, N-1\}$,

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} f\big(X(S)\big)ds + \int_{t_n}^{t_{n+1}} g\big(X(S)\big)dW_s$$

for $\in [t_n, t_{n+1}]$, the Taylor's expansion for f and g around $X(t_n)$ can be performed to obtain:

$$f\big(X(s)\big) = f\big(X(t_n)\big) + R_f\big(s; t_n, X(t_n)\big),$$

$$g\big(X(s)\big) = g\big(X(t_n)\big) + R_g\big(s; t_n, X(t_n)\big),$$

where R_f and R_g in the forms below:

$$R_f\big(s; t_n, X(t_n)\big) = f'(X^*)\big(X(s) - X(t_n)\big) \dots \qquad \dots (4)$$

$$R_g\big(s; t_n, X(t_n)\big) = g'(X^{**})\big(X(s) - X(t_n)\big) \dots \qquad \dots (5)$$

where $X^*, X^{**} \in [X(t_n) - X(t_{n+1})]$

From Eq. (3) we obtain:

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} f\big(X(t_n)\big)\, ds$$

$$+ \int_{t_n}^{t_{n+1}} g\big(X(t_n)\big)dW_s + \int_{t_n}^{t_{n+1}} R_f ds + \int_{t_n}^{t_{n+1}} R_g dW_s$$

If e_n is the error for the time t_n, and without losing of generality assume that $h \leq 1$:

$$e_{n+1} = X(t_{n+1}) - y_{n+1}$$

$$= X(t_n) - y_n + hf\big(X(t_n)\big) - hf(y_n) + \Delta W_{n+1} g(X(t_n)) - \Delta W_{n+1} g(y_n)$$

$$+ \int_{t_n}^{t_{n+1}} R_f ds + \int_{t_n}^{t_{n+1}} R_g dW_s$$

Let

$$\int_{t_n}^{t_{n+1}} R_f ds = \tilde{R}_f$$

$$\int_{t_n}^{t_{n+1}} R_g dW_s = \tilde{R}_g$$

then

$$e_{n+1} = e_n + h[f(X(t_n)) - f(y_n)] + \Delta W_{n+1}[g(X(t_n)) - g(y_n)] + \tilde{R}_f + \tilde{R}_g$$

Let $f(X(t_n)) - f(y_n) = a$ and $g(X(t_n)) - g(y_n) = b$, Since the function $g(X)$ is constant at $b = 0$, then:

$$e_{n+1} = e_n + ha + \tilde{R}_f + \tilde{R}_g$$

By squaring both hands of the above equation and taking the expectation, obtained:

$$E[{e_{n+1}}^2] = E\left[{e_n}^2 + h^2a^2 + \tilde{R}^2 + 2(ahe_n + \tilde{R}_f e_n + \tilde{R}_g e_n + ah\tilde{R})\right] \quad \ldots (6)$$

But the $E\left[\tilde{R}_g e_n\right]$ must be determined earlier by using “Tower Property”,

$$E\left[\tilde{R}_g e_n\right] = E\ \left[E\left[\tilde{R}_g e_n \middle| \mathcal{F}_{t_n}\right]\right]$$

$$= E\left[e_n\left[\tilde{R}_g|\mathcal{F}_{t_n}\right]\right] \qquad \dots (7)$$

Define :

$$M_n(u) = \int_{t_n}^{u} g'(X^{**})\left(X(s) - X(t_n)\right)dW_s \qquad \forall n \in \{0, \cdots, N\}$$

$M_n(u)$ is a $\mathcal{F}_{t_n}$-Martingale

$$\left[M_n(u)|\mathcal{F}_{t_n}\right] = M_n(t_n) \quad , \quad \forall u \geq t_n \qquad \dots (8)$$

and

$$M_n(t_n) = \int_{t_n}^{u} g'(X^{**})\left(X(s) - X(t_n)\right)dW_s \qquad \dots (9)$$

From Eq.'s (7), (8), (9),

$$E\left[\tilde{R}_g E_n\right] = E\ \left[e_n E\left[M_n(t_{n+1})|\mathcal{F}_{t_n}\right]\right]$$

$$=\ E\ [e_n.0] = 0 \qquad \dots (10)$$

By combining (6) and (10), yields:

$$E[{e_{n+1}}^2] = E\left[{e_n}^2 + h^2a^2 + \tilde{R}^2 + 2(ahe_n + \tilde{R}_f e_n + ah\tilde{R})\right] \qquad \dots (11)$$

Since f and g are proven to be global Lipschitz's functions, then $\exists\ K_1, K_2 \ni n \in \{0, \cdots, N\}$.

$$|a| =\ |f(X(t_n)) - f(y_n)| \leq K_1|X(t_n) - y_n|$$

$$|b| =\ |g(X(t_n)) - g(y_n)| \leq K_2|X(t_n) - y_n|$$

Let $K_3 = \max\ (K_1, K_2)$ such that,

$$|a|, |b| \leq K_3|X(t_n) - y_n| = K_3|E_n| \qquad \dots (12)$$

and $|a|^2, |b|^2 \leq {K_3}^2|e_n|^2$

By using the triangle inequalities in Eq. (11) yields,

$$E[|{e_{n+1}}^2|] = E\left[|e_n|^2 + h^2a^2 + \tilde{R}^2 + 2(|ahe_n| + \left|\tilde{R}_f e_n\right| + h\left|a\tilde{R}\right|)\right]$$

$$\leq E\left[|e_n|^2 + h^2{K_3}^2|e_n|^2 + \tilde{R}^2 + 2\left(K_3h|{e_n}^2| + \left|\tilde{R}_f e_n\right| + hK_3\left|e_n\tilde{R}\right|\right)\right] \dots (13)$$

From Eq. (4)

$$\left|\tilde{R}_f\right|^2 = \left|\int_{t_n}^{t_{n+1}} f'(X^*)\big(X(s) - X(t_n)\big)ds\right|^2$$

$$\leq \left|\int_{t_n}^{t_{n+1}} K_3\big(X(s) - X(t_n)\big)ds\right|^2$$

$$\leq {K_3}^2h\int_{t_n}^{t_{n+1}} |X(s) - X(t_n)|^2\, ds$$

By using the Cauchy-Schwarz's theorem and since (12) yields

$$f' \leq K_3 \ \ \forall X \in \mathbb{R}$$

Thus,

$$E\left[\left|\tilde{R}_f\right|^2\right] \leq {K_3}^2h \int_{t_n}^{t_{n+1}} E|X(s) - X(t_n)|^2\, ds \qquad \dots (14)$$

Now $[E|X(s) - X(t_n)|^2]$ will be calculated by using Chaucy-Schwarz's inequality, the linearity of expectation and Ito's isometry, lemma 1, lemma 2:

$$[E|X(s) - X(t_n)|^2] \leq 3\,E\left|\int_{t_n}^{s} f\big(X(r)\big)dr\right|^2 + 3E\left|\int_{t_n}^{s} g\big(X(r)\big)dW_r\right|^2$$

$$\leq 3(s - t_n)\int_{t_n}^{s} E\left[\left|f\big(X(r)\big)\right|^2\right]dr + 3\int_{t_n}^{s} E\left[\left|g\big(X(r)\big)\right|^2\right]dr$$

$$\leq 3\int_{t_n}^{s} E\left[\left|f\big(X(r)\big)\right|^2\right]dr + 3\int_{t_n}^{s} E\left[\left|g\big(X(r)\big)\right|^2\right]dr$$

$$\leq 3\int_{t_n}^{s} E[K_4(1 + |X(r)|^2)]dr + 3\int_{t_n}^{s} E[K_5(1 + |X(r)|^2)]dr$$

$$\leq 3(K_4 + K_5)\int_{t_n}^{s} E\left[1 + \sup_{r \in [t_n, s]} |X(r)|^2\right]$$

$$\leq 3(K_4 + K_5)\int_{t_n}^{s} K_6 dr$$

$$\leq K_7(s - t_n)$$

Hence

$$E\left[\left|\tilde{R}_f\right|^2\right] \leq {K_3}^2 h\int_{t_n}^{t_{n+1}} K_7(s - t_n)ds$$

$$\leq K_8 h\int_{t_n}^{t_{n+1}} K_7(t_{n+1} - t_n)ds$$

$$\leq K_8 h \int_{t_n}^{t_{n+1}} h\, ds \;\leq K_8 h^3$$

Furthermore, by using Itô's isometry,

$$E\left|\tilde{R}_g\right|^2 = E\left|\int_{t_n}^{t_{n+1}} g'(X^{**})\big(X(s) - X(t_n)\big)dWs\right|^2$$

$$= E\left|\int_{t_n}^{t_{n+1}} \left|g'(X^{**})\big(X(s) - X(t_n)\big)\right|^2 ds\right|$$

Since $g(X)$ is a constant function then $g'(X^{**}) = 0$, and hence $E\left|\tilde{R}_g\right|^2 = 0$

Given that:

$$\left|\tilde{R}\right|^2 = \left|\tilde{R}_f + \tilde{R}_g\right|^2 \leq 3\left|\tilde{R}_f\right|^2 + 3\left|\tilde{R}_g\right|^2$$

Since $h \leq 1$

$$E\left[\left|\tilde{R}\right|^2\right] \leq 3K_9 h^3 + 3(0) = K_{10} h^2$$

Observe that,

$$E\left[\left|\tilde{R}_f e_n\right|\right] = E\left[\left|\tilde{R}_f\right||e_n|\right]$$

$$= E\left[\left(\frac{1}{\sqrt{h}}\left|\tilde{R}_f\right|\right)\left(\sqrt{h}|e_n|\right)\right]$$

$$\leq E\left[\frac{1}{2}\left(\frac{1}{\sqrt{h}}\left|\tilde{R}_f\right|\right)^2 + \frac{1}{2}\left(\sqrt{h}|e_n|\right)^2\right]$$

$$= E\left[\frac{1}{2h}\left|\tilde{R}_f\right|^2 + \frac{h}{2}|e_n|^2\right]$$

then:

$$E[|e_{n+1}{}^2|] \leq E\left[|e_n|^2 + h^2K_3{}^2|e_n|^2 + \tilde{R}^2 + 2\left(K_1h|e_n|^2 + \left|\tilde{R}_f e_n\right| + hK_3\left|e_n\tilde{R}\right|\right)\right]$$

$$\leq E\left[|e_n|^2 + h^2K_3{}^2|e_n|^2 + K_{10}h^2 + 2\left(K_3h|e_n|^2 + \frac{1}{2h}K_8h^3 + \frac{h}{2}|e_n|^2 + \frac{1}{2}hK_3|e_n|^2 + \frac{1}{2}hK_3\left|\tilde{R}\right|^2\right)\right]$$

$$\leq E\left[|e_n|^2 + h^2K_3{}^2|e_n|^2 + K_{10}h^2 + 2\left(K_3h|e_n|^2 + \frac{K_8}{2}h^2 + \frac{h}{2}|e_n|^2 + \frac{1}{2}hK_3|e_n|^2 + \frac{1}{2}K_3K_{10}h^3\right)\right]$$

$$= E\left[|e_n|^2 + h^2K_3{}^2|e_n|^2 + K_{10}h^2 + 2K_3h|e_n|^2 + K_8h^2 + h|e_n|^2 + hK_3|e_n|^2 + K_3K_{10}h^3\right]$$

$$= E[|e_n|^2]\left(1 + K_3{}^2h^2 + 2K_3h + h + K_3h\right) + K_{10}h^2 + K_8h^2 + K_3K_{10}h^3$$

$$= E[|e_n|^2]\left(1 + K_3{}^2h^2 + 3K_3h + h\right) + K_3K_{10}h^3 + (K_{10} + K_8)h^2$$

$$= E[|e_n|^2](1 + Kh^2 + Kh) + Kh^3 + Kh^2$$

where, K=max $(K_3{}^2, 3K_3 + 1, K_3K_{10}, K_{10} + K_8)$

Then since $e_0 = 0$

$$E[|e_N|^2] = \sum_{i=0}^{N-1} E[|e_{i+1}|^2 - |e_i|^2]$$

$$\leq \sum_{i=0}^{N-1} \big(E[|e_i|^2](Kh^2 + Kh) + K(h^3 + h^2)\big)$$

$$= KT(h^2 + h) + \sum_{i=0}^{N-1} E[|e_i|^2](Kh^2 + Kh)$$

Now,

$$E[|e_{n+1}|^2 - |e_n|^2] \leq \ E[|e_n|^2](Kh^2 + Kh) + K(h^3 + h^2), \quad \forall n = 0, \dots, N-1$$

Therefore, by using Gronwall's discrete:

$$E[|e_N|^2] = KT(h^2 + h)\ exp\left(\sum_{i=0}^{N-1} (Kh^2 + Kh)\right)$$

$$= KT(h^2 + h)\ exp\ (KTh + KT$$

$$\leq 2KTh\ exp\ (2KT)$$

$$= Ch$$

where $C = 2KT\ exp(2KT)$

$$E[|e_N|^2] \leq Ch\,, \qquad \text{for } h \leq 1$$

Thus, obtained $E[|e_N|^2] \leq Ch$. According to the definition of convergence, the E-M method is proven to have strong convergence for solving the Ornstein-Uhlenbeck's equation.

C. Algorithm representing Euler-Maruyama's method for solving the Orstein-Uhlenbeck's equation

Based on the general form of the Euler-Maruyama method described above, an algorithm representing the steps taken in the method to find the numerical solution of the Ornstein-Uhlenbeck's equation can be arranged as follows:

Input : Two functions $f(X(t)) = \lambda(\mu - X(t))$ and $g(X(t)) = \sigma$, discretizing $[0, T]$ into N intervals of width $h = \frac{T}{N}$, and X_0 (initial value).

Output : A numerical solution for Ornstein-Uhlenbeck's equation.

Processes :

i. Set $dt := \frac{T}{N}$

ii. dW are random numbers normally distributed and set $W := cumsum(dW)$

iii. Set $Dt := R * dt$ and $L := \frac{N}{R}$ (step measures of E-M's method)

iv. Set $X_i := X_0$

v. For $i := 1$ to L :

Set $Winc := sum(dW(R * (j - 1) + 1: R * j));$

Set $X_{i+1} := X_i + f(X_i) * Dt + g(X_i) * Winc;$

End for.

Algorithm simulation

The following graphs obtained from Matlab™ program implementing the algorithm:

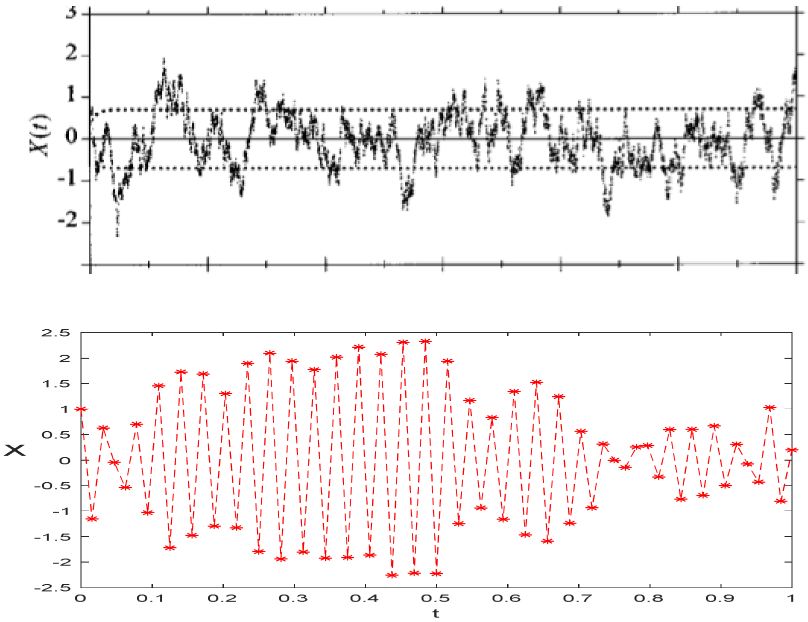
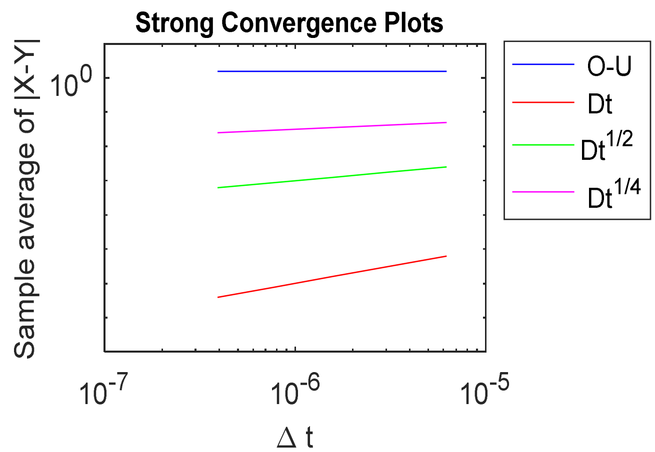


Figure 1. Charts for exact solution compared to numerical solutions obtained by E-M's method for Ornstein-Uhlenbeck's equation

Description:

The Ornstein-Uhlenbeck's equation: $2(1 - X(t))dt + 2dW(t)$

From both charts it is shown that the exact solution coincides with the approximate (numerical) solution obtained by using E-M's method. In other words, both solutions are said to be relatively close.



Fugure 2. Convergence plot for Euler-Maruyama's method for Ornstein-Uhlenbeck's equation.

Description:

In the Ornstein-Uhlenbeck's equation the fucntion f and g satisify Lipschitz's condition. It also has been shown that the E-M's method has strong convergence at $\gamma = \frac{1}{2}$. According to the graphic above, the smaller γ the stronger the E-M's approximate converges to the exact solution since the chart approach the Ornstein-Uhlenbeck's faster.

CHAPTER IV

SUMMARY

A. Conclusion

Based on the results discussed and the numerical simulation on the algorithm built, the following conclusions can be drawn from this research:

1. The formula of E-M's method for an Ornstein-Uhlenbeck's equation:

$$dX(t) = \lambda\big(\mu - X(t)\big)dt + \sigma dW(t)$$

is stated as follows:

$$X(t_{n+1}) = X(t_n) + \lambda\big(\mu - X(t_n)\big)(t_{n+1} - t_n) + \sigma(W(t_{n+1}) - W(t_n)),$$

where $n = 0, \dots, N$

2. The result shows that $E[|e_N|^2] \leq Ch$ then according to the definition of convergence, the E-M's method has a trong convergence to the exact solution of Ornstein-Uhlenbeck's equation.
3. The following algorithm represents the E-M's method for an Ornstein-Uhlenbeck's equation:

Input : Two functions $f(X(t)) = \lambda\big(\mu - X(t)\big)$ and $g\big(X(t)\big) = \sigma$, discretizing $[0, T]$ into N intervals of width $h = \frac{T}{N}$, and X_0 (initial value).

Output : A numerical solution for Ornstein-Uhlenbeck's equation.

Processes :

i. Set $dt \coloneqq \frac{T}{N}$

ii. dW are random numbers normally distributed and set $W \coloneqq cumsum(dW)$

iii. Set $Dt \coloneqq R * dt$ and $L \coloneqq \frac{N}{R}$ (step measures of E-M's method)

iv. Set $X_i \coloneqq X_0$

v. For $i \coloneqq 1$ to L :

Set $Winc \coloneqq sum(dW(R * (j-1) + 1 : R * j));$

Set $X_{i+1} \coloneqq X_i + f(X_i) * Dt + g(X_i) * Winc;$

End for.

B. Recommendation

This study only disusses the Euler-Maruyama’s method for finding the numerical solutions of the Ornstein-Uhlenbeck’s equation but have not yet discuss another numerical method for solving stochastic differential equations, namely the Euler-Milstein method. For this reason it is suggested thet further researchers discuss the Euler-Milstein method for the numerical solution of the Ornstein-Uhlenbeck equation, or compare the efficacy and the efficiency of both methods.

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APPENDICES

Appendix 1. Matlab™ program for the Euler-Maruyama's method

```
randn('state',100)
lambda=2; mu=1; Xzero=1; sig=2;
T=1; N=2^8; dt=1/N;
dW=sqrt(dt)*randn(1,N);
W=cumsum(dW);
Xeksak = mu+(Xzero-mu)*exp(-lambda*(dt:dt:T))+(sig*exp(-
         lambda*(dt:dt:T)))*((exp(lambda*(dt:dt:T))*W)-
         sum(exp(lambda*T([0,1:end-1])*W(T([0,1:end-1])))));
plot([0:dt:T],[Xzero,Xeksak],'m-'), hold on

R=4; Dt=R*dt; L=N/R;
Xem=zeros(1,L);
Xtemp=Xzero;
for j=1:L
    Winc=sum(dW(R*(j-1)+1:R*j));
    Xtemp=Xtemp+Dt*lambda*mu-lambda*Xtemp+sig*Winc;
    Xem(j)=Xtemp;
end
plot([0:Dt:T],[Xzero,Xem],'r--*'), hold off
xlabel('t','FontSize',12)
ylabel('X','FontSize',16,'Rotation',0,'HorizontalAlignment','right')
emerr=abs(Xem(end)-Xeksak(end))
```

Appendix 2. Matlab™ program for the strong convergence of Euler-Maruyama's method

```
tic;
clc;
clear all;
lambda=2;
mu=1;
Xzero=1;
sig=2;
T=1;
N=1000;

for q=1:1000
X(1)=0.75;
Y(1)=0.75;
t(1)=0;
s(1)=0;
dW(1) = sqrt(dt)*randn;
B(1) = dW(1);
for i = 2:N+1
    dW(i) = sqrt(dt)*randn;
    B(i) = B(i-1)+dW(i);
end
%Refference%
for i= 1:N
    f(i+1) = exp(lambda*t(i))*(dW(i+1)-dW(i));
    t(i+1) = t(i) + dt;
end
```

```
 Xeksak = mu+(Xzero-mu)*exp(-lambda*T)+sig*exp(-lambda*T)*sum(f);

for p =4:8
    Dt = 2^(-p)/10*dt;
    for i=1:N
       X(i+1) = X(i)+ Dt*(lambda*mu-lambda*X(i))+sig*dW(i+1);
    end
Xerr(q,p) = abs(X(N+1)-Xeksak);
end
end

for p=4:8
    Dtvals(p) = 2^(-p)/10*dt;
end
for p=4:8
    a=0;
    for q=1:1000
        a=a+(Xerr(q,p));
    end
    b(p)=a/1000;
end

subplot(221)
loglog(Dtvals,b,'b'); hold on;
loglog(Dtvals,(Dtvals).^(1),'r');hold on;
loglog(Dtvals,(Dtvals).^(0.5),'g');hold on;
loglog(Dtvals,(Dtvals).^(0.25),'m');hold off;
xlabel('\Delta t','FontSize',10)
ylabel('Sample average of |X-Y|','FontSize',10)
legend(' O-U',' Dt', 'Dt^{1/2}',' Dt^{1/4}');
title('Strong Convergence Plots ','FontSize',9)
% xticks([1e-7 1e-6 1e-5]);
% yticks([1e-8 1e-7 1e-6 1e-5 1e-4 1e-3 1e-2 1e-1]);
axis([1e-7 1e-5 1e-8 10]);
time=toc
```

Appendix 3. Proof for Ito's theorem

Step 1. Assume that $X(t)$ is finite, K is constant such that $V(x,t)$ for irrelevant $x \notin [-K,K]$.

Else, for each $n \geq 1$, the stop time is defined by

$$\tau_n = \inf\{t \geq 0: |x(t)| \geq n\}$$

Clearly that $\tau_n \uparrow \infty$. A stochastic process is also defined by

$$x_n(t) = [-n \vee x(0)] \wedge n + \int_0^t f(s) I_{[[0,\tau_n]]}(s) ds + \int_0^t g(s) I_{[[0,\tau_n]]}(s) dB_s$$

At $t \geq 0$, then $|x_n(t)| \leq n$, then $x_n(t)$ is finite. In fact, for each $t \geq 0$ and $\omega \in \Omega$, where there exists $n_o = n_o(t, \omega)$ such that

$$x_n(s, \omega) = x(s, \omega) \qquad \text{where } 0 \leq s \leq t$$

Take that $n \geq n_o$, an Ito's formula $x_n(t)$ is obtained i.e:

$$V(x_n(t), t) - V(x(0), 0)$$
$$= \int_0^t \Big[V_t(x_n(s), s) + V_x(x_n(s), s) f(s) I_{[[0,\tau_n]]}(s)$$
$$+ \frac{1}{2} V_{xx}(x_n(s), s) g^2(s) I_{[[0,\tau_n]]}(s) \Big] ds + \int_0^t V_x(x_n(s), s) g(s) I_{[[0,\tau_n]]}(s) dB_s$$

After letting $n \to \infty$, then the desired result is obtained

Step 2.

Assume that $V(x, t)$ along with C^2 have the second order derivatives with respect to (x, t), Furthermore, the sequence $\{V_n(x, t)\}$ can be obtained from the function C^2 such that

$$\{V_n(x, t)\} \to V(x, t), \qquad \frac{\partial}{\partial t} V_n(x, t) \to V_t(x, t)$$

$$\frac{\partial}{\partial x} V_n(x, t) \to V_x(x, t), \qquad \frac{\partial^2}{\partial x^2} V_n(x, t) \to V_{xx}(x, t)$$

The Ito's formula can be shown for each V_n, namely

$$V(x_n(t), t) - V(x(0), 0)$$
$$= \int_0^t \left[\frac{\partial}{\partial x} V_n(x(s), s) + \frac{\partial}{\partial x} V_n(x(s), s) f(s) + \frac{1}{2} \frac{\partial^2}{\partial x^2} V_n(x(s), s) g^2(s) \right] ds$$
$$+ \int_0^t \frac{\partial}{\partial x} V_n(x(s), s) g(s) dB_s$$

Then by letting $n \to \infty$, the desired result is obtained. Step 1 and 2 are assumed without loss of generality, that is $V, V_t, V_{tt}, V_x, V_{tx}$ and V_{xx} are finite in $R \times [0, t]$ for each $t \geq 0$.

Step 3.

The Ito's formula can also be shown where f and g are simple processes. Then the general cases are followed by approximate solution

Step 4.

Set any $t > 0$ and assume that $V, V_t, V_{tt}, V_x, V_{tx}, V_{xx}$ are finite in $R \times [0, t]$ and $f(s), g(s)$ are two simple processes at $s \in [0, t]$. Let $\Pi = \{t_0, t_1, \cdots t_k\}$ be the partition of $[0, t]$. $f(s)$ and $g(s)$ are random constant in $(t_i, t_{i+1}]$, means that

$$f(s) = f_i, \quad g(s) = g_i$$

Using Taylor's expansion, obtained that

$$V(x(t), t) - V(x(0), 0)$$

$$= \sum_{i=0}^{k-1} [V(x(t_{i+1}), t_{i+1}) - V(x(t_i), t_i)]$$

$$= \sum_{i=0}^{k-1} V_t(x(t_i), t_i)\Delta t_i + \sum_{i=0}^{k-1} V_x(x(t_i), t_i)\Delta x_i + \frac{1}{2}\sum_{i=0}^{k-1} V_{tt}(x(t_i), t_i)(\Delta t_i)^2$$

$$+ \sum_{i=0}^{k-1} V_{tx}(x(t_i), t_i)\Delta t_i \Delta x_i + \frac{1}{2}\sum_{i=0}^{k-1} V_{xx}(x(t_i), t_i)(\Delta x_i)^2 + \sum_{i=0}^{k-1} R_i$$

where

$$\Delta t_i = t_{i+1} - t_i \ , \qquad \Delta x_i = x(t_{i+1}) - x(t_i) \ , \ R_i = o(((\Delta t_i)^2) + ((\Delta x_i)^2))$$

Set $|\Pi| = max_{0 \leq t \leq k-1} \Delta t_i$ .it is quite easy to see that $|\Pi| \to 0$, with probability of 1,

$$\sum_{i=0}^{k-1} V_t(x(t_i), t_i)\Delta t_i \to \int_0^t V_t(x(s), s)ds$$

$$\sum_{i=0}^{k-1} V_x(x(t_i), t_i)\Delta x_i \to \int_0^t V_x(x(s), s)dx(s) = \int_0^t V_x(x(s), s)f(s)ds + \int_0^t V_x(x(s), s)g(s)dB_s$$

$$\sum_{i=0}^{k-1} V_{tt}(x(t_i), t_i)(\Delta t_i)^2 \to 0 \qquad \sum_{i=0}^{k-1} R_i \to 0$$

$$\sum_{i=0}^{k-1} V_{tx}(x(t_i), t_i)\Delta t_i \Delta x_i = \sum_{i=0}^{k-1} V_{tx}(x(t_i), t_i) f_i (\Delta x_i)^2 + \sum_{i=0}^{k-1} V_{tx}(x(t_i), t_i)\Delta t_i \Delta B_i$$

where $\Delta B_i = B_{t_{i+1}} - B_{t_i}$, when $|\Pi| \to 0$, the first term tends to 0 while the second term tends to 0 in L^2 hence

$$E\left(\sum_{i=0}^{k-1} V_{tx}(x(t_i), t_i) g_i \Delta t_i \Delta B_i\right)^2 = \sum_{i=0}^{k-1} [V_{tx}(x(t_i), t_i) g_i]^2 \, (\Delta t_i)^2 \to 0$$

In other words,

$$\sum_{i=0}^{k-1} V_{tx}(x(t_i), t_i)\Delta t_i x_i \to 0, \quad dalam\ L^2$$

Observe that

$$\sum_{i=0}^{k-1} V_{xx}(x(t_i), t_i)(\Delta x_i)^2$$

$$= \sum_{i=0}^{k-1} V_{xx}(x(t_i), t_i)\left[{f_i}^2(\Delta t_i)^2 + 2 f_i g_i \Delta t_i \Delta B_i\right] + \sum_{i=0}^{k-1} V_{xx}(x(t_i), t_i){g_i}^2(\Delta B_i)^2$$

The first term tends to 0 in L^2 as $|\Pi| \to 0$ for the same reason, meanwhile the second term tends to $\int_0^t V_{xx}(x(s),s)\, g^2(s)ds$ in L^2. To show further, set $h(t) = \int_0^t V_{xx}(x(t),t)\, g^2(t)$, $h_i = V_{xx}(x(t_i),t_i){g_i}^2(\Delta B_i)^2$

$$E\left(\sum_{i=0}^{k-1} h_i(\Delta B_i)^2 + \sum_{i=0}^{k-1} h_i(\Delta t_i)^2\right)^2 = E\left(\sum_{i=0}^{k-1}\sum_{i=0}^{k-1} h_i h_j[(\Delta B_i)^2 - \Delta t_i]\left[(\Delta B_i)^2 - \Delta t_j\right]\right)$$

$$= \sum_{i=0}^{k-1} E\left({h_i}^2[(\Delta B_i)^2 - \Delta t_i]^2\right)$$

$$= \sum_{i=0}^{k-1} E{h_i}^2 E[(\Delta B_i)^4 - 2((\Delta B_i)^2)\Delta t_i + (\Delta t_i)^2]$$

$$= \sum_{i=0}^{k-1} E{h_i}^2[3(\Delta t_i)^2 - 2((\Delta t_i)^2) + (\Delta t_i)^2]$$

$$= 2\sum_{i=0}^{k-1} E{h_i}^2\,(\Delta t_i)^2 \to 0$$

Where we have used $(\Delta B_i)^{2n} = (2n)!\;(\Delta t_i)^n/2^n n!$. Thus

$$\sum_{i=0}^{k-1} E{h_i}^2\,(\Delta B_i)^2 \to \int_0^t h(s)ds \quad in \quad L^2$$

Substituting Eq 5-9 to Eq. 4 yields

$$V(x(t),t) - V(x(0),0)$$

$$= \int_0^t \left[V_t(x(s),s) + V_x(x(s),s)f(s) + \frac{1}{2}V_{xx}(x(s),s)g^2(s)\right]ds$$

$$+ \int_0^t V_x(x(s),s)g(s)dB_s$$

PROVEN.

Appendix 4. Proof for Taylor's series theorem

The following is the proof of Taylor's series theorem with an integral residual term. Basic calculus theorem tells that

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

can be rearranged as

$$f(x) = f(a) + \int_a^x f'(t)\,dt \qquad (1)$$

Then look at the partial integral form

$$\int_{v=a}^b u\,dv = uv|_{v=a}^b - \int_{u=a}^b v\,du$$

$$\int_{v=a}^b u\,dv = u(b-a) - \int_{u=a}^b v\,du$$

By applied the partial integral on the second term in the right hand of $\int_{t=a}^x f'(t)\,dt$ in Eq. (1) then it is assumed that

$$u = f'(t) \rightarrow du = f''(t)\,dt$$

$$v = t \rightarrow dv = dt$$

thus

$$\int_{t=a}^x f'(t)\,dt = f'(t)(t)|_a^x - \int_{t=a}^x t\,f''(t)\,dt$$

$$\int_{t=a}^x f'(t)\,dt = xf'(x) - af'(a) - \int_{t=a}^x t\,f''(t)\,dt \qquad (2)$$

Substituting Eq. (2) into Eq. (1) yields

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

$$= f(a) + xf'(x) - af'(a) - \int_a^x t\,f''(t)\,dt \qquad (3)$$

Observe that $\int_{t=a}^x x\,f''(t)\,dt = xf'(x) - xf'(a)$ or it can be written that

$xf'(x) = \int_{t=a}^x xf''(t)\,dt + xf'(a)$, such that Eq. (3) becomes

$$f(x) = f(a) + \int_a^x x\,f''(t)\,dt + xf'(a) - af'(a) - \int_{t=a}^x t\,f''(t)\,dt$$

$$= f(a) + xf'(a) - af'(a) + \int_a^x x\,f''(t)\,dt - \int_{t=a}^x t\,f''(t)\,dt$$

$$= f(a) + f'(a)(x-a) + \int_{t=a}^{x}(x-t)\, f''(t)\, dt \tag{4}$$

Reuse the partial integral on $\int_{t=a}^{x}(x-t)f''(t)\, dt$ in the Eq (4) by letting

$$u = f''(t) \quad \rightarrow \quad du = f'''(t)\, dt$$

$$dv = (x-t)dt \quad \rightarrow \quad v = xt - \frac{t^2}{2}$$

Obtained that

$$\int_{t=a}^{x}(x-t)f''(t)\, dt = \; f''(t)\, \left(x^2 - \frac{x^2}{2}\right)\Big|_{t=a}^{x} - \int_{t=a}^{x}\left(xt - \frac{t^2}{2}\right)f'''(t)\, dt$$

$$\int_{t=a}^{x}(x-t)f''(t)\, dt = f''(x)\, \left(x^2 - \frac{x^2}{2}\right) - f''(a)\, \left(ax - \frac{a^2}{2}\right) -$$

$$\int_{t=a}^{x}(x^2 - \frac{x^2}{2})f'''(t)\, dt \tag{5}$$

Substituting Eq. (5) into (4) yields

$$f(x) = f(a) + f'(a)(x-a) + \int_{t=a}^{x}(x-t)\, f''(t)\, dt$$

$$= f(a) + f'(a)(x-a) + f''(x)\, \left(x^2 - \frac{x^2}{2}\right) - f''(a)\, \left(ax - \frac{a^2}{2}\right) -$$

$$\int_{t=a}^{x}(x^2 - \frac{x^2}{2})f'''(t)\, dt$$

$$= f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(x)\,(x-a)^2 + \tfrac{1}{2}\int_{a}^{x}(x-t)^2 f'''(t)\, dt$$

If this process is repeated for n times, then it will be obtained a series called the **Taylor's series:**

$$f(x) = f(a) + \; f'(a)(x-a) + \frac{f''(x)}{2!}(x-a)^2 + \frac{f'''(x)}{3!}(x-a)^3 + \; \ldots$$

$$+ \frac{f^{(n)}(x)}{n!}(x-a)^n + \; R_n(x)$$

where $R_n(x) = \frac{1}{n!}\int_{a}^{x} f^{(n+1)}(t)(x-t)^n\, dt$, $R_n(x)$ is the residual or error term for $f(x)$.